

MULTIPLIERS OF HILBERT PRO- C^* -BIMODULES AND CROSSED PRODUCTS BY HILBERT PRO- C^* -BIMODULES

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ABSTRACT. In this paper we introduce the notion of multiplier of a Hilbert pro- C^* -bimodule and we investigate the structure of the multiplier bimodule of a Hilbert pro- C^* -bimodule. We also investigate the relationship between the crossed product $A \times_X \mathbb{Z}$ of a pro- C^* -algebra A by a Hilbert pro- C^* -bimodule X over A , the crossed product $M(A) \times_{M(X)} \mathbb{Z}$ of the multiplier algebra $M(A)$ of A by the multiplier bimodule $M(X)$ of X and the multiplier algebra $M(A \times_X \mathbb{Z})$ of $A \times_X \mathbb{Z}$.

1. INTRODUCTION

The notion of a Hilbert C^* -module is a generalization of that of a Hilbert space in which the inner product takes its values in a C^* -algebra rather than in the field of complex numbers, but the theory of Hilbert C^* -modules is different from the theory of Hilbert spaces (for example, no every Hilbert C^* -submodule is complemented). In 1953, Kaplansky first used Hilbert C^* -modules over commutative C^* -algebras to prove that derivations of type I AW^* -algebras are inner. In 1973, the theory was extended independently by Paschke and Rieffel to non-commutative C^* -algebras and the latter author used it to construct the theory of “induced representations of C^* -algebras”. Moreover, Hilbert C^* -modules gave the right context for the extension of the notion of Morita equivalence to C^* -algebras and have played a crucial role in Kasparov’s KK -theory. Finally, they may be considered as a generalization of vector bundles to non-commutative $*$ -algebras, therefore they play a significant role in non-commutative geometry and, in particular, in C^* -algebraic quantum group theory and groupoid C^* -algebras. The extension of such a rich in results concept, to the case of pro- C^* -algebras could not be disregarded.

In [Z], Zarakas introduced the notion of a Hilbert pro- C^* -bimodule over a pro- C^* -algebra and studied its structure. In [J2], JoiȚa investigated the structure of the multiplier module of a Hilbert pro- C^* -module. In this paper we introduce the notion of multiplier of a Hilbert pro- C^* -bimodule and we investigate the structure of the multiplier bimodule of a Hilbert pro- C^* -bimodule.

In [JZ1], JoiȚa and Zarakas extended the construction of Abadie, Eilers and Exel [AEE] in context of pro- C^* -algebras and associated to a Hilbert pro- C^* -bimodule (X, A) a pro- C^* -algebra $A \times_X \mathbb{Z}$, called the crossed product of A by X . It is natural to ask what is the relationship between the pro- C^* -algebras associated to a Hilbert pro- C^* -bimodule (X, A) and its multiplier bimodule $(M(X), M(A))$.

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The organization of this paper is as follows. In Section 2, we recall some notations and definitions. Section 3 is devoted to investigate multipliers of a Hilbert pro- C^* -bimodule. Given a Hilbert pro- C^* -bimodule X , we show that the Hilbert pro- C^* -bimodule structure on X extends to a Hilbert pro- C^* -bimodule structure on the multiplier bimodule $M(X)$ of X . Also we define the strict topology on $M(X)$ and show that X can be identified with a Hilbert pro- C^* -sub-bimodule of $M(X)$ which is dense in $M(X)$ with respect to the strict topology. We introduce the notion of morphism of Hilbert pro- C^* -bimodules, and show that a nondegenerate morphism between Hilbert pro- C^* -bimodules is continuous with respect to the strict topology and it extends to a unique morphism between the multiplier bimodules. Finally, as in the case of Hilbert C^* -bimodules [R], we show that $(M(X), M(A))$ can be regarded as a maximal extension of (X, A) . Section 4 is devoted to investigate the relationship between the crossed product $A \times_X \mathbb{Z}$ of a pro- C^* -algebra A by a Hilbert pro- C^* -bimodule X over A , the crossed product $M(A) \times_{M(X)} \mathbb{Z}$ of the multiplier algebra $M(A)$ of A by the multiplier bimodule $M(X)$ of X and the multiplier algebra $M(A \times_X \mathbb{Z})$ of $A \times_X \mathbb{Z}$. We show that the crossed product associated to a full Hilbert pro- C^* -bimodule (X, A) can be identified with a pro- C^* -subalgebra of the crossed product associated to $(M(X), M(A))$ and the crossed product associated to $(M(X), M(A))$ can be identified with a pro- C^* -subalgebra of the multiplier algebra of the crossed product associated to (X, A) . Crossed products by Hilbert pro- C^* -bimodules are generalizations of crossed products of pro- C^* -algebras by inverse limit automorphism [JZ1]. As an application, we prove that given an inverse limit automorphism α of a nonunital pro- C^* -algebra A , the crossed product of $M(A)$ by $\bar{\alpha}$, the extension of α to $M(A)$, can be identified with a pro- C^* -subalgebra of the multiplier algebra $M(A \times_\alpha \mathbb{Z})$ of $A \times_\alpha \mathbb{Z}$.

2. PRELIMINARIES

A complete Hausdorff topological $*$ -algebra A whose topology is given by a directed family of C^* -seminorms $\{p_\lambda; \lambda \in \Lambda\}$ is called a *pro- C^* -algebra*. Other terms used in the literature for pro- C^* -algebras are: locally C^* -algebras (A. Inoue, M. Fragoulopoulou, A. Mallios, etc.), *LMC * -algebras* (G. Lassner, K. Schmüdgen), *b * -algebras* (C. Apostol).

Let A be a pro- C^* -algebra with the topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$ and let B be a pro- C^* -algebra with the topology given by $\Gamma' = \{q_\delta; \delta \in \Delta\}$.

An *approximate unit* of A is a net $\{e_i\}_{i \in I}$ of positive elements in A such that $p_\lambda(e_i) \leq 1$ for all $i \in I$ and for all $\lambda \in \Lambda$ and the nets $\{e_i b\}_{i \in I}$ and $\{b e_i\}_{i \in I}$ converge to b for all $b \in A$.

A *pro- C^* -morphism* is a continuous $*$ -morphism $\varphi : A \rightarrow B$ (that is, φ is linear, $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ for all $a, b \in A$ and for each $q_\delta \in \Gamma'$, there is $p_\lambda \in \Gamma$ such that $q_\delta(\varphi(a)) \leq p_\lambda(a)$ for all $a \in A$). An invertible pro- C^* -morphism $\varphi : A \rightarrow B$ is a pro- C^* -isomorphism if φ^{-1} is also pro- C^* -morphism.

If $\{A_\lambda; \pi_{\lambda\mu}\}_{\lambda \geq \mu, \lambda, \mu \in \Lambda}$ is an inverse system of C^* -algebras, then $\lim_{\leftarrow \lambda} A_\lambda$ with the topology given by the family of C^* -seminorms $\{p_\lambda\}_{\lambda \in \Lambda}$, with $p_\lambda \left((a_\mu)_{\mu \in \Lambda} \right) = \|a_\lambda\|_{A_\lambda}$ for all $\lambda \in \Lambda$, is a pro- C^* -algebra.

Let A be a pro- C^* -algebra with the topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$. For $\lambda \in \Lambda$, $\ker p_\lambda$ is a closed $*$ -bilateral ideal and $A_\lambda = A / \ker p_\lambda$ is a C^* -algebra in the C^* -norm $\|\cdot\|_{p_\lambda}$ induced by p_λ (that is, $\|a + \ker p_\lambda\|_{p_\lambda} = p_\lambda(a)$, for all $a \in A$). The

canonical map from A to A_λ is denoted by π_λ^A , $\pi_\lambda^A(a) = a + \ker p_\lambda$ for all $a \in A$. For $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$ there is a surjective C^* -morphism $\pi_{\lambda\mu}^A : A_\lambda \rightarrow A_\mu$ such that $\pi_{\lambda\mu}^A(a + \ker p_\lambda) = a + \ker p_\mu$, and then $\{A_\lambda; \pi_{\lambda\mu}^A\}_{\lambda, \mu \in \Lambda}$ is an inverse system of C^* -algebras. Moreover, the pro- C^* -algebras A and $\lim_{\leftarrow \lambda} A_\lambda$ are isomorphic (Arens-Michael decomposition). For more details we refer the reader to [F, M, P].

Here we recall some basic facts from [J1] and [Z] regarding Hilbert pro- C^* -modules and Hilbert pro- C^* -bimodules respectively.

Let A be a pro- C^* -algebra whose topology is given by the family of C^* -seminorms $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$.

A *right Hilbert pro- C^* -module over A* (or just *Hilbert A -module*), is a linear space X that is also a right A -module equipped with a right A -valued inner product $\langle \cdot, \cdot \rangle_A$, that is \mathbb{C} - and A -linear in the second variable and conjugate linear in the first variable, with the following properties:

- (1) $\langle x, x \rangle_A \geq 0$ and $\langle x, x \rangle_A = 0$ if and only if $x = 0$;
- (2) $(\langle x, y \rangle_A)^* = \langle y, x \rangle_A$

and which is complete with respect to the topology given by the family of seminorms $\{p_\lambda^A\}_{\lambda \in \Lambda}$, with $p_\lambda^A(x) = p_\lambda(\langle x, x \rangle_A)^{\frac{1}{2}}$, $x \in X$. A Hilbert A -module X is full if the pro- C^* -subalgebra of A generated by $\{\langle x, y \rangle_A; x, y \in X\}$ coincides with A .

A *left Hilbert pro- C^* -module X* over a pro- C^* -algebra A is defined in the same way, where for instance the completeness is requested with respect to the family of seminorms $\{^A p_\lambda\}_{\lambda \in \Lambda}$, where $^A p_\lambda(x) = p_\lambda(^A \langle x, x \rangle)^{\frac{1}{2}}$, $x \in X$.

In case X is a left Hilbert pro- C^* -module over $(A, \{p_\lambda\}_{\lambda \in \Lambda})$ and a right Hilbert pro- C^* -module over $(B, \{q_\lambda\}_{\lambda \in \Lambda})$, such that the following relations hold:

- $^A \langle x, y \rangle z = x \langle y, z \rangle_B$ for all $x, y, z \in X$,
- $q_\lambda^B(ax) \leq p_\lambda(a)q_\lambda^B(x)$ and $^A p_\lambda(xb) \leq q_\lambda(b)^A p_\lambda(x)$ for all $x \in X$, $a \in A$, $b \in B$ and for all $\lambda \in \Lambda$,

then we say that X is a *Hilbert $A - B$ pro- C^* -bimodule*.

A Hilbert $A - B$ pro- C^* -bimodule X is *full* if it is full as a right and as a left Hilbert pro- C^* -module.

Let Λ be an upward directed set and $\{A_\lambda; B_\lambda; X_\lambda; \pi_{\lambda\mu}; \chi_{\lambda\mu}; \sigma_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ an inverse system of Hilbert C^* -bimodules, that is:

- $\{A_\lambda; \pi_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ and $\{B_\lambda; \chi_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ are inverse systems of C^* -algebras;
- $\{X_\lambda; \sigma_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ is an inverse system of Banach spaces;
- for each $\lambda \in \Lambda$, X_λ is a Hilbert $A_\lambda - B_\lambda$ C^* -bimodule;
- $\langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y) \rangle_{B_\mu} = \chi_{\lambda\mu}(\langle x, y \rangle_{B_\lambda})$ and $_{A_\mu} \langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y) \rangle = \pi_{\lambda\mu}(_{A_\lambda} \langle x, y \rangle)$ for all $x, y \in X_\lambda$ and for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$.
- $\sigma_{\lambda\mu}(x)\chi_{\lambda\mu}(b) = \sigma_{\lambda\mu}(xb)$, $\pi_{\lambda\mu}(a)\sigma_{\lambda\mu}(x) = \sigma_{\lambda\mu}(ax)$ for all $x \in X_\lambda$, $a \in A_\lambda$, $b \in B_\lambda$ and for all $\lambda, \mu \in \Lambda$ such that $\lambda \geq \mu$.

Let $A = \lim_{\leftarrow \lambda} A_\lambda$, $B = \lim_{\leftarrow \lambda} B_\lambda$ and $X = \lim_{\leftarrow \lambda} X_\lambda$. Then X has a structure of a Hilbert $A - B$ pro- C^* -bimodule with

$$\begin{aligned} (x_\lambda)_{\lambda \in \Lambda} (b_\lambda)_{\lambda \in \Lambda} &= (x_\lambda b_\lambda)_{\lambda \in \Lambda} \text{ and } \langle (x_\lambda)_{\lambda \in \Lambda}, (y_\lambda)_{\lambda \in \Lambda} \rangle_B = (\langle x_\lambda, y_\lambda \rangle_{B_\lambda})_{\lambda \in \Lambda} \\ &\text{and} \\ (a_\lambda)_{\lambda \in \Lambda} (x_\lambda)_{\lambda \in \Lambda} &= (a_\lambda x_\lambda)_{\lambda \in \Lambda} \text{ and } _A \langle (x_\lambda)_{\lambda \in \Lambda}, (y_\lambda)_{\lambda \in \Lambda} \rangle = (_{A_\lambda} \langle x_\lambda, y_\lambda \rangle)_{\lambda \in \Lambda}. \end{aligned}$$

Let X be a Hilbert $A - B$ pro- C^* -bimodule. Then, for each $\lambda \in \Lambda$, ${}^A p_\lambda(x) = q_\lambda^B(x)$ for all $x \in X$, and the normed space $X_\lambda = X/N_\lambda^B$, where $N_\lambda^B = \{x \in X; q_\lambda^B(x) = 0\}$, is complete in the norm $\|x + N_\lambda^B\|_{X_\lambda} = q_\lambda^B(x)$, $x \in X$. Moreover, X_λ has a canonical structure of a Hilbert $A_\lambda - B_\lambda$ C^* -bimodule with $\langle x + N_\lambda^B, y + N_\lambda^B \rangle_{B_\lambda} = \langle x, y \rangle_B + \ker q_\lambda$ and ${}_{A_\lambda} \langle x + N_\lambda^B, y + N_\lambda^B \rangle = {}_A \langle x, y \rangle + \ker p_\lambda$ for all $x, y \in X$. The canonical surjection from X to X_λ is denoted by σ_λ^X . For $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, there is a canonical surjective linear map $\sigma_{\lambda\mu}^X : X_\lambda \rightarrow X_\mu$ such that $\sigma_{\lambda\mu}^X(x + N_\lambda^B) = x + N_\mu^B$ for all $x \in X$. Then $\{A_\lambda; B_\lambda; X_\lambda; \pi_{\lambda\mu}^A; \sigma_{\lambda\mu}^X; \pi_{\lambda\mu}^B; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ is an inverse system of Hilbert C^* -bimodules in the above sense.

Let X and Y be Hilbert pro- C^* -modules over B . A morphism $T : X \rightarrow Y$ of right modules is *adjointable* if there is another morphism of modules $T^* : Y \rightarrow X$ such that $\langle Tx, y \rangle_B = \langle x, T^*y \rangle_B$ for all $x \in X, y \in Y$. The vector space $L_B(X, Y)$ of all adjointable module morphisms from X to Y has a structure of locally convex space under the topology given by the family of seminorms $\{q_{\lambda, L_B(X, Y)}\}_{\lambda \in \Lambda}$, where $q_{\lambda, L_B(X, Y)}(T) = \sup\{q_\lambda^B(Tx); x \in X, q_\lambda^B(x) \leq 1\}$. Moreover, $\{L_{B_\lambda}(X_\lambda, Y_\lambda); \chi_{\lambda\mu}^{L_B(X, Y)}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ where $\chi_{\lambda\mu}^{L_B(X, Y)} : L_{B_\lambda}(X_\lambda, Y_\lambda) \rightarrow L_{B_\mu}(X_\mu, Y_\mu)$ is given by $\chi_{\lambda\mu}^{L_B(X, Y)}(T)(\sigma_\mu^X(x)) = \sigma_\mu^Y(T(\sigma_\lambda^X(x)))$, is an inverse system of Banach spaces and $L_B(X, Y) = \varprojlim_{\lambda \in \Lambda} L_{B_\lambda}(X_\lambda, Y_\lambda)$ up to an isomorphism of locally convex spaces. The canonical projections $\chi_\lambda^{L_B(X, Y)} : L_B(X, Y) \rightarrow L_{B_\lambda}(X_\lambda, Y_\lambda)$, $\lambda \in \Lambda$ are given by $\chi_\lambda^{L_B(X, Y)}(T)(\sigma_\lambda^X(x)) = \sigma_\lambda^Y(T(x))$ for all $x \in X$. For $x \in X$ and $y \in Y$, the map $\theta_{y, x} : X \rightarrow Y$ given by $\theta_{y, x}(z) = y \langle x, z \rangle_B$ is an adjointable module morphism and the closed subspace of $L_B(X, Y)$ generated by $\{\theta_{y, x}; x \in X \text{ and } y \in Y\}$ is denoted by $K_B(X, Y)$, whose elements are usually called *compact operators*. For $Y = X$, $L_B(X) = L_B(X, X)$ is a pro- C^* -algebra with $(L_B(X))_\lambda = L_{B_\lambda}(X_\lambda)$ for each $\lambda \in \Lambda$, and $K_B(X) = K_B(X, X)$ is a closed two-sided $*$ -ideal of $L_B(X)$ with $(K_B(X))_\lambda = K_{B_\lambda}(X_\lambda)$ for each $\lambda \in \Lambda$.

A pro- C^* -algebra A has a natural structure of Hilbert pro- C^* -module, and the multiplier algebra $M(A)$ has a structure of pro- C^* -algebra which is isomorphic to $L_A(A)$ [P]. Moreover, pro- C^* -algebras A and $K_A(A)$ are isomorphic and A is a closed bilateral ideal of $M(A)$ which is dense in $M(A)$ with respect to the strict topology. The strict topology on $M(A)$ is given by the family of seminorms $\{p_{(\lambda, a)}\}_{(\lambda, a) \in \Lambda \times A}$, where $p_{(\lambda, a)}(b) = p_\lambda(ab) + p_\lambda(ba)$ for all $b \in M(A)$.

A pro- C^* -morphism $\varphi : A \rightarrow M(B)$ is nondegenerate if $[\varphi(A)B] = B$, where $[\varphi(A)B]$ denotes the closed subspace of B generated by $\{\varphi(a)b; a \in A, b \in B\}$. A nondegenerate pro- C^* -morphism $\varphi : A \rightarrow M(B)$ extends to a unique pro- C^* -morphism $\overline{\varphi} : M(A) \rightarrow M(B)$ which is strictly continuous on bounded sets.

Throughout this paper, A and B are two pro- C^* -algebras whose topologies are given by the families of C^* -seminorms $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$, respectively $\Gamma' = \{q_\delta; \delta \in \Delta\}$.

3. MULTIPLIERS OF HILBERT PRO- C^* -BIMODULES

Let X and Y be two Hilbert pro- C^* -modules over A .

Proposition 3.1. *The vector space $L_A(X, Y)$ of all adjointable module maps from X to Y has a natural structure of Hilbert $L_A(Y) - L_A(X)$ pro- C^* -bimodule with the bimodule structure given by*

$$S \cdot T = S \circ T \text{ and } T \cdot R = T \circ R$$

for all $T \in L_A(X, Y)$, $S \in L_A(Y)$ and $R \in L_A(X)$ and the inner products given by

$${}_{L_A(Y)} \langle T_1, T_2 \rangle = T_1 \circ T_2^* \text{ and } \langle T_1, T_2 \rangle_{L_A(X)} = T_1^* \circ T_2$$

for all $T_1, T_2 \in L_A(X, Y)$.

Proof. It is a simple calculation to verify that $L_A(X, Y)$ has a structure of pre-right Hilbert $L_A(X)$ -pro- C^* -module with

$$T \cdot R = T \circ R \text{ and } \langle T_1, T_2 \rangle_{L_A(X)} = T_1^* \circ T_2$$

and $L_A(X, Y)$ has a structure of pre-left Hilbert $L_A(Y)$ -pro- C^* -module with

$$S \cdot T = S \circ T \text{ and } {}_{L_A(Y)} \langle T_1, T_2 \rangle = T_1 \circ T_2^*.$$

Moreover,

$$\begin{aligned} p_\lambda^{L_A(X)}(T)^2 &= p_{\lambda, L_A(X)} \left(\langle T, T \rangle_{L_A(X)} \right) = p_{\lambda, L_A(X)}(T^* \circ T) \\ &= \left\| \chi_\lambda^{L_A(X, Y)}(T)^* \chi_\lambda^{L_A(X, Y)}(T) \right\|_{L_{A_\lambda}(X_\lambda)} \\ &\quad (\text{see, for example, the proof of Proposition 1.10 [EKQR]}) \\ &= \left\| \chi_\lambda^{L_A(X, Y)}(T) \right\|_{L_{A_\lambda}(X_\lambda, Y_\lambda)}^2 = p_{\lambda, L_A(X, Y)}(T)^2 \end{aligned}$$

and

$$\begin{aligned} {}^{L_A(Y)} p_\lambda(T)^2 &= p_{\lambda, L_A(Y)} \left({}_{L_A(Y)} \langle T, T \rangle \right) = p_{\lambda, L_A(Y)}(T \circ T^*) \\ &= \left\| \chi_\lambda^{L_A(X, Y)}(T) \chi_\lambda^{L_A(X, Y)}(T)^* \right\|_{L_{A_\lambda}(Y_\lambda)} \\ &= \left\| \chi_\lambda^{L_A(X, Y)}(T)^* \right\|_{L_{A_\lambda}(Y_\lambda, X_\lambda)}^2 \\ &\quad (\text{see, for example, the proof of Proposition 1.10 [EKQR]}) \\ &= \left\| \chi_\lambda^{L_A(X, Y)}(T) \right\|_{L_{A_\lambda}(X_\lambda, Y_\lambda)}^2 = p_{\lambda, L_A(X, Y)}(T)^2 \end{aligned}$$

for all $T \in L_A(X, Y)$ and for all $\lambda \in \Lambda$. Therefore, $L_A(X, Y)$ is a left Hilbert $L_A(Y)$ -module and a right Hilbert $L_A(X)$ -module.

Also it is easy to check that ${}_{L_A(Y)} \langle T_1, T_2 \rangle \cdot T_3 = T_1 \cdot \langle T_2, T_3 \rangle_{L_A(X)}$ for all $T_1, T_2, T_3 \in L_A(X, Y)$, and since $p_\lambda^{L_A(X)}(T) = {}^{L_A(Y)} p_\lambda(T) = p_{\lambda, L_A(X, Y)}(T)$ for all $T \in L_A(X, Y)$ and for all $\lambda \in \Lambda$, $L_A(X, Y)$ has a structure of Hilbert $L_A(Y)$ - $L_A(X)$ pro- C^* -bimodule. \square

Remark 3.2. Suppose that (X, A) is a full Hilbert pro- C^* -bimodule. Then there is a pro- C^* -isomorphism $\Phi_A : A \rightarrow K_A(X)$ given by $\Phi_A(a)(x) = a \cdot x$ which extends to a pro- C^* -isomorphism $\overline{\Phi_A} : M(A) \rightarrow L_A(X)$. Moreover, $p_{\lambda, L_A(X)}(\Phi_A(a)) = p_\lambda(a)$ for all $a \in A$ and $\lambda \in \Lambda$. Identifying $M(A)$ with $L_A(A)$ and using Proposition 3.1 and [R, Proposition 2.5], we obtain a natural structure of Hilbert $M(A)$ - $M(A)$ pro- C^* -bimodule on $L_A(A, X)$ with

$$m \cdot T = \overline{\Phi_A}(m) \circ T \text{ and } {}_{M(A)} \langle T_1, T_2 \rangle = \overline{\Phi_A^{-1}}(T_1 \circ T_2^*)$$

and

$$T \cdot m = T \circ m \text{ and } \langle T_1, T_2 \rangle_{M(A)} = T_1^* \circ T_2$$

for all $T, T_1, T_2 \in L_A(A, X)$ and $m \in M(A)$.

Definition 3.3. Let (X, A) be a full Hilbert pro- C^* -bimodule. The Hilbert $M(A) - M(A)$ pro- C^* -bimodule $L_A(A, X)$ is called the multiplier bimodule of X and it is denoted by $M(X)$.

The following definition is a generalization of [EKQR, Definition 1.25].

Definition 3.4. The strict topology on $M(X)$ is given by the family of seminorms $\{p_{(\lambda, a)}\}_{(\lambda, a) \in \Lambda \times A}$, where $p_{(\lambda, a)}(T) = p_\lambda^{M(A)}(T \cdot a) + p_\lambda^{M(A)}(a \cdot T)$ for all $T \in M(X)$ and $a \in A$.

Remark 3.5. Let $\{T_n\}_n$ be a sequence in $M(X)$.

- (1) If $\{T_n\}_n$ is strictly convergent, then it is bounded. Indeed, if $\{T_n\}_n$ converges strictly to T , then for each $\lambda \in \Lambda$, since

$$\begin{aligned} \left\| \chi_\lambda^{M(X)}(T_n) \pi_\lambda^A(a) - \chi_\lambda^{M(X)}(T) \pi_\lambda^A(a) \right\|_{X_\lambda} &= p_\lambda^A(T_n(a) - T(a)) \\ &= p_\lambda^{M(A)}(T_n \cdot a - T \cdot a), \end{aligned}$$

the sequence $\{\chi_\lambda^{M(X)}(T_n) \pi_\lambda^A(a)\}_n$ converges to $\chi_\lambda^{M(X)}(T) \pi_\lambda^A(a)$ for all $a \in A$ and by the Banach-Steinhaus theorem there is $M_\lambda > 0$ such that

$$p_\lambda^{M(A)}(T_n) = p_{\lambda, L_A(A, X)}(T_n) = \left\| \chi_\lambda^{M(X)}(T_n) \right\|_{L_{A_\lambda}(A_\lambda, X_\lambda)} \leq M_\lambda.$$

- (2) If $\{T_n\}_n$ converges strictly to 0, then the sequences $\{\langle T_n, T_n \rangle_{M(A)}\}_n$ and $\{\langle T_n, T_n \rangle_{M(A)}\}_n$ are strictly convergent to 0 in $M(A)$.

Suppose that X is a Hilbert pro- C^* -module over A . In [J2, Definition 3.2], the strict topology on $L_A(A, X)$ is given by the family of seminorms $\{p_{(\lambda, a, x)}\}_{(\lambda, a, x) \in \Lambda \times A \times X}$, where $p_{(\lambda, a, x)}(T) = p_\lambda^A(T(a)) + p_\lambda(T^*(x))$. We will show that this definition coincides with the above definition of the strict topology on $M(X)$ on bounded subsets when X is a full Hilbert $A - A$ pro- C^* -bimodule. To show this, we will use the following result.

Lemma 3.6. Let X be a Hilbert pro- C^* -module over A . For each x in X there is a unique element y in X such that $x = y \langle y, y \rangle_A$.

Proof. Let $x \in X$. For each $\lambda \in \Lambda$, there is a unique element $y_\lambda \in X_\lambda$ such that $\sigma_\lambda^X(x) = y_\lambda \langle y_\lambda, y_\lambda \rangle_{A_\lambda}$ (see, for example, [RW, Proposition 2.31]). Let $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$. From

$$\sigma_\mu^X(x) = \sigma_{\lambda\mu}^X(\sigma_\lambda^X(x)) = \sigma_{\lambda\mu}^X(y_\lambda) \langle \sigma_{\lambda\mu}^X(y_\lambda), \sigma_{\lambda\mu}^X(y_\lambda) \rangle_{A_\mu}$$

and [RW, Proposition 2.31], we deduce that $\sigma_{\lambda\mu}^X(y_\lambda) = y_\mu$. Therefore, there exists $y \in X$ such that $\sigma_\lambda^X(y) = y_\lambda$ for all $\lambda \in \Lambda$ and $x = y \langle y, y \rangle_A$. Moreover, y is unique with this property. \square

Proposition 3.7. Let (X, A) be a full Hilbert pro- C^* -bimodule and $\{T_i\}_{i \in I}$ a net in $M(X)$.

- (1) If $\{T_i\}_{i \in I}$ converges strictly to 0, then $\{p_{(\lambda, a, x)}(T_i)\}_{i \in I}$ converges to 0 for all $a \in A$, for all $x \in X$ and for all $\lambda \in \Lambda$.
- (2) If $\{T_i\}_{i \in I}$ is bounded and $\{p_{(\lambda, a, x)}(T_i)\}_{i \in I}$ converges to 0 for all $a \in A$, for all $x \in X$ and for all $\lambda \in \Lambda$, then $\{T_i\}_{i \in I}$ converges strictly to 0.

Proof. (1) If the net $\{T_i\}_{i \in I}$ converges strictly to 0, then $\{p_\lambda^A(T_i(a))\}_{i \in I}$ converges to 0 for all $a \in A$ and $\lambda \in \Lambda$. Let $x \in X$ and $\lambda \in \Lambda$. Then, by Lemma 3.6, there is $y \in X$ such that $x = y \langle y, y \rangle_A = \theta_{y,y}(y)$. From

$$\begin{aligned} p_\lambda(T_i^*(x)) &= p_\lambda(T_i^*(\theta_{y,y}(y))) \leq p_{\lambda, L_A(X,A)}(T_i^* \circ \theta_{y,y}) p_\lambda^A(y) \\ &= p_{\lambda, L_A(A,X)}(\theta_{y,y} \circ T_i) p_\lambda^A(y) = p_\lambda^{M(A)}(\theta_{y,y} \circ T_i) p_\lambda^A(y), \end{aligned}$$

we deduce that the net $\{p_\lambda(T_i^*(x))\}_{i \in I}$ converges to 0.

(2) If $\{p_{(\lambda,a,x)}(T_i)\}_{i \in I}$ converges to 0 for all $a \in A$, $x \in X$ and $\lambda \in \Lambda$, then $\{p_\lambda^A(T_i(a))\}_{i \in I}$ converges to 0 for all $a \in A$ and $\lambda \in \Lambda$. Let $S \in K_A(X)$, $\lambda \in \Lambda$ and $\varepsilon > 0$. Then there is $\sum_{k=1}^n \theta_{x_k, y_k}$ such that $p_{\lambda, L_A(X)}\left(S - \sum_{k=1}^n \theta_{x_k, y_k}\right) < \varepsilon$, and since $\{T_i\}_{i \in I}$ is bounded, there is $M_\lambda > 0$ such that $p_\lambda^{M(A)}(T_i) \leq M_\lambda$ for all $i \in I$. From

$$\begin{aligned} p_\lambda^{M(A)}(S \circ T_i) &\leq p_{\lambda, L_A(X)}\left(S - \sum_{k=1}^n \theta_{x_k, y_k}\right) p_\lambda^{M(A)}(T_i) + p_\lambda^{M(A)}\left(\sum_{k=1}^n \theta_{x_k, y_k} \circ T_i\right) \\ &\leq \varepsilon M_\lambda + p_{\lambda, L_A(A,X)}\left(\sum_{k=1}^n \theta_{x_k, T_i^*(y_k)}\right) \\ &\leq \varepsilon M_\lambda + \sum_{k=1}^n p_\lambda^A(x_k) p_\lambda(T_i^*(y_k)) \end{aligned}$$

we deduce that $\{p_\lambda^{M(A)}(S \circ T_i)\}_{i \in I}$ converges to 0. \square

Let (X, A) and (Y, B) be two Hilbert pro- C^* -bimodules.

Definition 3.8. A morphism of Hilbert pro- C^* -bimodules from (X, A) to (Y, B) is a pair (Φ, φ) consisting of a pro- C^* -morphism $\varphi : A \rightarrow B$ and a map $\Phi : X \rightarrow Y$ such that:

- (1) $\Phi(xa) = \Phi(x) \varphi(a)$ for all $x \in X$ and for all $a \in A$;
- (2) $\Phi(ax) = \varphi(a) \Phi(x)$ for all $x \in X$ and for all $a \in A$;
- (3) $\langle \Phi(x), \Phi(y) \rangle_B = \varphi(\langle x, y \rangle_A)$ for all $x, y \in X$;
- (4) ${}_B \langle \Phi(x), \Phi(y) \rangle = \varphi({}_A \langle x, y \rangle)$ for all $x, y \in X$.

The relation (3) implies the relation (1) and the relation (4) implies (2).

If $(\Phi, \varphi) : (X, A) \rightarrow (Y, B)$ is a morphism of Hilbert pro- C^* -bimodules, then Φ is continuous, since for each $\delta \in \Delta$, there is $\lambda \in \Lambda$ such that

$$q_\delta^B(\Phi(x))^2 = q_\delta(\langle \Phi(x), \Phi(x) \rangle_B) = q_\delta(\varphi(\langle x, x \rangle_A)) \leq p_\lambda(\langle x, x \rangle_A) = p_\lambda^A(x)^2$$

for all $x \in X$. It is easy to check that if φ is injective, then Φ is injective, and if (X, A) is full and Φ is injective, then φ is injective.

Definition 3.9. An isomorphism of Hilbert pro- C^* -bimodules is a morphism of Hilbert pro- C^* -bimodules (Φ, φ) such that φ is a pro- C^* -isomorphism and the map Φ is bijective.

Hilbert pro- C^* -bimodules (X, A) and (Y, B) are isomorphic if there is an isomorphism of Hilbert pro- C^* -bimodules $(\Phi, \varphi) : (X, A) \rightarrow (Y, B)$.

Definition 3.10. A morphism of Hilbert pro- C^* -bimodules $(\Phi, \varphi) : (X, A) \rightarrow (M(Y), M(B))$ is nondegenerate if φ is nondegenerate and $[\Phi(X)B] = Y$.

Remark 3.11. If $(\Phi, \varphi) : (X, A) \rightarrow (M(Y), M(B))$ is nondegenerate and (X, A) is full, then (Φ, φ) is nondegenerate in the sense of [J3, Definition 3.1], since

$$\begin{aligned} [\Phi(X)^* Y] &= [\Phi(X)^* \Phi(X) B] = [\langle \Phi(X), \Phi(X) \rangle_{M(B)} B] \\ &= [\varphi(\langle X, X \rangle_A) B] = [\varphi(A) B] = B. \end{aligned}$$

Lemma 3.12. Let (X, A) be a full Hilbert pro- C^* -bimodule. Then the maps

$$\left(\chi_{\lambda}^{L_A(A, X)}, \pi_{\lambda}^{M(A)} \right) : (M(X), M(A)) \rightarrow (M(X_{\lambda}), M(A_{\lambda})), \lambda \in \Lambda,$$

where $\pi_{\lambda}^{M(A)} = \chi_{\lambda}^{L_A(A)}$, and

$$\left(\chi_{\lambda\mu}^{L_A(A, X)}, \pi_{\lambda\mu}^{M(A)} \right) : (M(X_{\lambda}), M(A_{\lambda})) \rightarrow (M(X_{\mu}), M(A_{\mu})), \lambda, \mu \in \Lambda \text{ with } \lambda \geq \mu$$

where $\pi_{\lambda\mu}^{M(A)} = \chi_{\lambda\mu}^{L_A(A)}$, are all strictly continuous morphisms of Hilbert bimodules.

Proof. Let $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$. For $T_1, T_2 \in M(X_{\lambda})$ we have

$$\begin{aligned} \left\langle \chi_{\lambda\mu}^{L_A(A, X)}(T_1), \chi_{\lambda\mu}^{L_A(A, X)}(T_2) \right\rangle_{M(A_{\mu})} &= \chi_{\lambda\mu}^{L_A(A, X)}(T_1)^* \circ \chi_{\lambda\mu}^{L_A(A, X)}(T_2) \\ &= \chi_{\lambda\mu}^{L_A(X, A)}(T_1^*) \circ \chi_{\lambda\mu}^{L_A(A, X)}(T_2) \\ &= \chi_{\lambda\mu}^{L_A(A)}(T_1^* \circ T_2) = \pi_{\lambda\mu}^{M(A)}(\langle T_1, T_2 \rangle_{M(A_{\lambda})}) \end{aligned}$$

and

$$\begin{aligned} \left\langle \chi_{\lambda\mu}^{L_A(A, X)}(T_1), \chi_{\lambda\mu}^{L_A(A, X)}(T_2) \right\rangle_{M(A_{\mu})} &= \overline{\Phi_{A_{\mu}}^{-1}} \left(\chi_{\lambda\mu}^{L_A(A, X)}(T_1) \circ \chi_{\lambda\mu}^{L_A(A, X)}(T_2)^* \right) \\ &= \overline{\Phi_{A_{\mu}}^{-1}} \left(\chi_{\lambda\mu}^{L_A(A, X)}(T_1) \circ \chi_{\lambda\mu}^{L_A(X, A)}(T_2^*) \right) \\ &= \overline{\Phi_{A_{\mu}}^{-1}} \left(\chi_{\lambda\mu}^{L_A(X)}(T_1 \circ T_2^*) \right) \\ &= \chi_{\lambda\mu}^{L_A(A)} \left(\overline{\Phi_{A_{\lambda}}^{-1}}(T_1 \circ T_2^*) \right) \\ &= \pi_{\lambda\mu}^{M(A)}(M(A_{\lambda}) \langle T_1, T_2 \rangle). \end{aligned}$$

Therefore, $\left(\chi_{\lambda\mu}^{L_A(A, X)}, \pi_{\lambda\mu}^{M(A)} \right)$ is a morphism of Hilbert C^* -bimodules.

Let $\{T_i\}_{i \in I}$ be a net in $M(X_{\lambda})$ which converges strictly to 0. From

$$\begin{aligned} \left\| \chi_{\lambda\mu}^{L_A(A, X)}(T_i) \pi_{\mu}^A(a) \right\|_{X_{\mu}} &= \left\| \chi_{\lambda\mu}^{L_A(A, X)}(T_i) \pi_{\lambda\mu}^{M(A)}(\pi_{\lambda}^A(a)) \right\|_{X_{\mu}} \\ &= \left\| \sigma_{\lambda\mu}^X(T_i(\pi_{\lambda}^A(a))) \right\|_{X_{\mu}} \leq \|T_i(\pi_{\lambda}^A(a))\|_{X_{\lambda}} \end{aligned}$$

for all $a \in A$, and

$$\begin{aligned} \left\| \chi_{\mu}^{L_A(X)}(S) \circ \chi_{\lambda\mu}^{L_A(A, X)}(T_i) \right\|_{M(X_{\mu})} &= \left\| \chi_{\lambda\mu}^{L_A(X)} \left(\chi_{\lambda}^{L_A(X)}(S) \right) \circ \chi_{\lambda\mu}^{L_A(A, X)}(T_i) \right\|_{L_{A_{\mu}}(A_{\mu}, X_{\mu})} \\ &= \left\| \chi_{\lambda\mu}^{L_A(A, X)} \left(\chi_{\lambda}^{L_A(X)}(S) \circ T_i \right) \right\|_{L_{A_{\mu}}(A_{\mu}, X_{\mu})} \\ &\leq \left\| \chi_{\lambda}^{L_A(X)}(S) \circ T_i \right\|_{L_{A_{\lambda}}(A_{\lambda}, X_{\lambda})} \end{aligned}$$

for all $S \in K_A(X)$, and taking into account that $K_{A_{\mu}}(X_{\mu}) = \chi_{\mu}^{L_A(X)}(K_A(X))$, we deduce that the net $\{\chi_{\lambda\mu}^{M(X)}(T_i)\}_{i \in I}$ converges strictly to 0.

In a similar way, we show that the maps $\left(\chi_\lambda^{L_A(A,X)}, \pi_\lambda^{M(A)}\right) : (M(X), M(A)) \rightarrow (M(X_\lambda), M(A_\lambda)), \lambda \in \Lambda$ are all strictly continuous morphisms of Hilbert bimodules. \square

Theorem 3.13. *Let (X, A) be a full Hilbert pro- C^* -bimodule.*

- (1) $M(X)$ is complete with respect to the strict topology;
- (2) $(\iota_X, \iota_A) : (X, A) \rightarrow (M(X), M(A))$, where $\iota_X(x)(a) = xa$ and $\iota_A(b)(a) = ba$ for all $x \in X$ and $a, b \in A$, is a nondegenerate morphism of Hilbert pro- C^* -bimodules;
- (3) X can be identified with a closed $M(A) - M(A)$ pro- C^* -sub-bimodule of $M(X)$ which is dense in $M(X)$ with respect to the strict topology.

Proof. (1) For each $\lambda \in \Lambda$, $M(X_\lambda)$ has a structure of Hilbert $M(A_\lambda) - M(A_\lambda)$ C^* -bimodule (see, [EKQR, Proposition 1.10]). It is easy to check that

$$\{M(A_\lambda); M(X_\lambda); \pi_{\lambda\mu}^{M(A)}; \chi_{\lambda\mu}^{M(X)}; \lambda, \mu \in \Lambda, \lambda \geq \mu\},$$

where $\chi_{\lambda\mu}^{M(X)} = \chi_{\lambda\mu}^{L_A(A,X)}$ for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, is an inverse system of Hilbert C^* -bimodules. Then $\lim_{\leftarrow \lambda} M(X_\lambda)$ has a structure of Hilbert $\lim_{\leftarrow \lambda} M(A_\lambda) - \lim_{\leftarrow \lambda} M(A_\lambda)$ pro- C^* -bimodule. Moreover, by Lemma 3.12 the maps $\chi_{\lambda\mu}^{M(X)} : M(X_\lambda) \rightarrow M(X_\mu)$, $\lambda, \mu \in \Lambda, \lambda \geq \mu$ are all strictly continuous.

Consider, the maps:

$$\Phi : M(X) \rightarrow \lim_{\leftarrow \lambda} M(X_\lambda), \Phi(T) = \left(\chi_\lambda^{M(X)}(T)\right)_\lambda$$

and

$$\varphi : M(A) \rightarrow \lim_{\leftarrow \lambda} M(A_\lambda), \varphi(m) = \left(\pi_\lambda^{M(A)}(m)\right)_\lambda.$$

It is easy to check that (Φ, φ) is a morphism of Hilbert pro- C^* -bimodules. Moreover, Φ is bijective, and since φ is a pro- C^* -isomorphism, (Φ, φ) is an isomorphism of Hilbert pro- C^* -bimodules. Clearly, a net $\{T_i\}_{i \in I}$ in $M(X)$ converges strictly to 0 in $M(X)$ if and only if the net $\{\Phi(T_i)\}_{i \in I}$ converges strictly to 0 in $\lim_{\leftarrow \lambda} M(X_\lambda)$.

Therefore, the strict topology on $M(X)$ can be identified with the inverse limit of the strict topologies on $M(X_\lambda)$, $\lambda \in \Lambda$, and since $M(X_\lambda)$, $\lambda \in \Lambda$, are complete with respect to the strict topology [EKQR, Proposition 1.27], $M(X)$ is complete with respect to the strict topology.

(2) Let $\lambda \in \Lambda$. By [EKQR], $(\iota_{X_\lambda}, \iota_{A_\lambda}) : (X_\lambda, A_\lambda) \rightarrow (M(X_\lambda), M(A_\lambda))$, where $\iota_{X_\lambda}(\sigma_\lambda^X(x))(\pi_\lambda^A(a)) = \sigma_\lambda^X(xa)$ and $\iota_{A_\lambda}(\pi_\lambda^A(b))(\pi_\lambda^A(a)) = \pi_\lambda^A(ba)$ for all $x \in X$ and $a, b \in A$, is a morphism of Hilbert C^* -bimodules. Since

$$\chi_{\lambda\mu}^{M(X)} \circ \iota_{X_\lambda} = \iota_{X_\mu} \circ \sigma_{\lambda\mu}^X \text{ and } \pi_{\lambda\mu}^{M(A)} \circ \iota_{A_\lambda} = \iota_{A_\mu} \circ \pi_{\lambda\mu}^A$$

for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, there is a morphism of Hilbert pro- C^* -bimodules

$$\left(\lim_{\leftarrow \lambda} \iota_{X_\lambda}, \lim_{\leftarrow \lambda} \iota_{A_\lambda}\right) : \left(\lim_{\leftarrow \lambda} X_\lambda, \lim_{\leftarrow \lambda} A_\lambda\right) \rightarrow \left(\lim_{\leftarrow \lambda} M(X_\lambda), \lim_{\leftarrow \lambda} M(A_\lambda)\right).$$

Identifying X with $\lim_{\leftarrow \lambda} X_\lambda$ and A with $\lim_{\leftarrow \lambda} A_\lambda$, and using (1), we obtain a morphism of Hilbert pro- C^* -bimodules $(\iota_X, \iota_A) : (X, A) \rightarrow (M(X), M(A))$, where $\iota_X(x)(a) = xa$ and $\iota_A(b)(a) = ba$ for all $x \in X$ and $a, b \in A$. We know that ι_A is nondegenerate and XA is dense in X , and then (ι_X, ι_A) is nondegenerate.

(3) Since, for each $\lambda \in \Lambda$,

$$p_\lambda^{M(A)}(\iota_X(x)) = \|\iota_{X_\lambda}(\sigma_\lambda^X(x))\|_{M(X_\lambda)} = \|\sigma_\lambda^X(x)\|_{X_\lambda} = p_\lambda^A(x)$$

for all $x \in X$, X can be identified with a closed $M(A) - M(A)$ pro- C^* -sub-bimodule of $M(X)$. Using (1) – (2) and [M, Chapter III, Theorem 3.1], we have

$$\begin{aligned} \overline{\iota_X(X)}^{str} &= \overline{\lim_{\leftarrow \lambda} \chi_\lambda^{M(X)}(\iota_X(X))}^{str} = \overline{\lim_{\leftarrow \lambda} \iota_{X_\lambda}(\sigma_\lambda^X(X))}^{str} = \overline{\lim_{\leftarrow \lambda} \iota_{X_\lambda}(X_\lambda)}^{str} \\ &= \lim_{\leftarrow \lambda} M(X_\lambda) = M(X), \end{aligned}$$

where $\overline{}^{str}$ denotes the closure with respect to the strict topology of the Hilbert sub-bimodule Z of a Hilbert bimodule Y . Therefore, X can be identified with a closed $M(A) - M(A)$ pro- C^* -sub-bimodule of $M(X)$ which is dense in $M(X)$ with respect to the strict topology. \square

Remark 3.14. Let (X, A) be a full Hilbert pro- C^* -bimodule.

- (1) A net $\{x_i\}_{i \in I}$ in X converges strictly to 0 if and only if the nets $\{p_\lambda^A(x_i a)\}_{i \in I}$ and $\{p_\lambda^A(a x_i)\}_{i \in I}$ converge to 0 for all $a \in A$ and $\lambda \in \Lambda$.
- (2) The morphism of Hilbert pro- C^* -bimodules $(\iota_X, \iota_A) : (X, A) \rightarrow (M(X), M(A))$ is strictly continuous.

Lemma 3.15. Let (X, A) be a full Hilbert pro- C^* -bimodule, let $\{e_i\}_{i \in I}$ be an approximate unit for A and $T \in M(X)$. Then the net $\{T \cdot e_i\}_{i \in I}$ converges strictly to T .

Proof. The net $\{T \cdot e_i\}_{i \in I}$ is bounded, since

$$p_\lambda^{M(A)}(T \cdot e_i) \leq p_\lambda^{M(A)}(T) p_{\lambda, L_A(A)}(e_i) = p_\lambda^{M(A)}(T) p_\lambda(e_i) \leq p_\lambda^{M(A)}(T)$$

for all $i \in I$ and for $\lambda \in \Lambda$. From

$$p_\lambda^{M(A)}((T \cdot e_i - T)(a)) = p_\lambda^A(T(e_i a - a)) \leq p_{\lambda, L_A(A, X)}(T) p_\lambda(e_i a - a)$$

for all $a \in A$, $i \in I$, $\lambda \in \Lambda$, and

$$p_\lambda(((T \cdot e_i)^* - T^*)(x)) = p_\lambda(e_i T^*(x) - T^*(x))$$

for all $x \in X$, $i \in I$, $\lambda \in \Lambda$, and Proposition 3.7, and taking into account that $\{e_i\}_{i \in I}$ is an approximate unit for A , we conclude that $\{T \cdot e_i\}_{i \in I}$ converges strictly to T . \square

In the following theorem we show that any nondegenerate morphism of pro- C^* -bimodules is strictly continuous.

Theorem 3.16. Let (X, A) and (Y, B) be two full Hilbert pro- C^* -bimodules and let (Φ, φ) be a nondegenerate morphism of Hilbert pro- C^* -bimodules from (X, A) to $(M(Y), M(B))$. Then (Φ, φ) extends to a unique nondegenerate morphism of Hilbert pro- C^* -bimodules $(\overline{\Phi}, \overline{\varphi})$ from $(M(X), M(A))$ to $(M(Y), M(B))$. Moreover, $\overline{\Phi}$ is strictly continuous.

Proof. For each $\delta \in \Delta$, there is $\lambda \in \Lambda$ such that $q_{\delta, M(B)}(\varphi(a)) \leq p_\lambda(a)$ for all $a \in A$ and $q_\delta^{M(B)}(\Phi(x)) \leq p_\lambda^A(x)$ for all $x \in X$. So there exists a C^* -morphism $\varphi_{(\lambda, \delta)} : A_\lambda \rightarrow M(B_\delta)$ such that $\varphi_{(\lambda, \delta)} \circ \pi_\lambda^A = \pi_\delta^{M(B)} \circ \varphi$ and a linear map $\Phi_{(\lambda, \delta)} : X_\lambda \rightarrow M(Y_\delta)$ such that $\Phi_{(\lambda, \delta)} \circ \sigma_\lambda^X = \chi_\delta^{M(Y)} \circ \Phi$. It is easy to check that $(\Phi_{(\lambda, \delta)}, \varphi_{(\lambda, \delta)})$ is

a morphism of Hilbert C^* -bimodules from (X_λ, A_λ) to $(M(Y_\delta), M(B_\delta))$. Moreover, $(\Phi_{(\lambda, \delta)}, \varphi_{(\lambda, \delta)})$ is nondegenerate, since

$$[\varphi_{(\lambda, \delta)}(A_\lambda) B_\delta] = [\varphi_{(\lambda, \delta)}(\pi_\lambda^A(A)) B_\delta] = [\pi_\delta^{M(B)}(\varphi(A) B)] = [\pi_\delta^{M(B)}(B)] = B_\delta$$

and

$$\begin{aligned} [\Phi_{(\lambda, \delta)}(X_\lambda) B_\delta] &= [\Phi_{(\lambda, \delta)}(\sigma_\lambda^X(X)) \pi_\delta^{M(B)}(B)] = [\chi_\delta^{M(Y)}(\Phi(X) B)] \\ &= [\sigma_\delta^Y(Y)] = Y_\delta. \end{aligned}$$

Then, by [EKQR, Theorem 1.30], $\Phi_{(\lambda, \delta)}$ is strictly continuous and $(\Phi_{(\lambda, \delta)}, \varphi_{(\lambda, \delta)})$ extends to a unique nondegenerate morphism of Hilbert C^* -modules $(\overline{\Phi_{(\lambda, \delta)}}, \overline{\varphi_{(\lambda, \delta)}})$ from $(M(X_\lambda), M(A_\lambda))$ to $(M(Y_\delta), M(B_\delta))$. Let $\overline{\Phi_\delta} = \overline{\Phi_{(\lambda, \delta)}} \circ \chi_\lambda^{M(X)}$ and $\overline{\varphi_\delta} = \overline{\varphi_{(\lambda, \delta)}} \circ \pi_\lambda^{M(A)}$. Clearly, $(\overline{\Phi_\delta}, \overline{\varphi_\delta})$ is a morphism of pro- C^* -bimodules from $(M(X), M(A))$ to $(M(Y_\delta), M(B_\delta))$. Moreover, $\overline{\Phi_\delta}$ is strictly continuous, since $\chi_\lambda^{M(X)}$ is strictly continuous (see Lemma 3.12).

Let $\delta_1, \delta_2 \in \Delta$ with $\delta_1 \geq \delta_2$. We have

$$\begin{aligned} \overline{\Phi_{\delta_1}}(\iota_X(x)) &= (\overline{\Phi_{(\lambda_1, \delta_1)}} \circ \chi_{\lambda_1}^{M(X)})(\iota_X(x)) = \overline{\Phi_{(\lambda_1, \delta_1)}}(\iota_{X_{\lambda_1}}(\sigma_{\lambda_1}^X(x))) \\ &= \Phi_{(\lambda_1, \delta_1)}(\sigma_{\lambda_1}^X(x)) = \chi_{\delta_1}^{M(Y)}(\Phi(x)) \end{aligned}$$

for some $\lambda_1 \in \Lambda$ and for all $x \in X$. Then

$$(\chi_{\delta_1 \delta_2}^{M(Y)} \circ \overline{\Phi_{\delta_1}})(\iota_X(x)) = \chi_{\delta_1 \delta_2}^{M(Y)}(\chi_{\delta_1}^{M(Y)}(\Phi(x))) = \chi_{\delta_2}^{M(Y)}(\Phi(x)) = \overline{\Phi_{\delta_2}}(\iota_X(x))$$

for all $x \in X$. From these relations and taking into account that $\chi_{\delta_1 \delta_2}^{M(Y)}$, $\overline{\Phi_{\delta_1}}$, $\overline{\Phi_{\delta_2}}$ are strictly continuous and X is dense in $M(X)$ with respect to the strict topology, we conclude that $\chi_{\delta_1 \delta_2}^{M(Y)} \circ \overline{\Phi_{\delta_1}} = \overline{\Phi_{\delta_2}}$. Therefore there is a strictly continuous linear map $\overline{\Phi} : M(X) \rightarrow M(Y)$ such that $\chi_\delta^{M(Y)} \circ \overline{\Phi} = \overline{\Phi_\delta}$ for all $\delta \in \Delta$, and $\overline{\Phi} \circ \iota_X = \Phi$.

By [P, Proposition 3.15], there is a pro- C^* -morphism $\overline{\varphi} : M(A) \rightarrow M(B)$ such that $\pi_\delta^{M(B)} \circ \overline{\varphi} = \overline{\varphi_\delta}$ for all $\delta \in \Delta$ and $\overline{\varphi} \circ \iota_A = \varphi$.

It is easy to check that $(\overline{\Phi}, \overline{\varphi})$ is a morphism of Hilbert pro- C^* -bimodules. Since $\overline{\varphi}$ is nondegenerate [J1, Proposition 6.1.4] and

$$\begin{aligned} [\overline{\Phi}(M(X)) B] &= [\overline{\Phi}(M(X)) \varphi(A) B] = [\overline{\Phi}(M(X) A) B] \\ &= [\Phi(X) B] = Y \end{aligned}$$

the morphism of Hilbert pro- C^* -bimodule $(\overline{\Phi}, \overline{\varphi})$ is nondegenerate.

Suppose that there is another morphism of Hilbert pro- C^* -bimodules $(\Phi_1, \varphi_1) : (M(X), M(A)) \rightarrow (M(Y), M(B))$ such that $\Phi_1(\iota_X(x)) = \Phi(x)$ for all $x \in X$ and $\varphi_1(\iota_A(a)) = \varphi(a)$ for all $a \in A$. Let $\{e_i\}_{i \in I}$ be a approximate unit for A . Then, by Lemma 3.15 for each $T \in M(X)$ and $m \in M(A)$, the nets $\{T \cdot e_i\}_{i \in I}$ and $\{m \cdot e_i\}_{i \in I}$ are strictly convergent to T respectively m . Thus we have

$$\Phi_1(T) = \text{str-}\lim_i \Phi_1(T \cdot e_i) = \text{str-}\lim_i \Phi(T \cdot e_i) = \overline{\Phi}(T)$$

for all $T \in M(X)$ and

$$\varphi_1(m) = \text{str-}\lim_i \varphi_1(m \cdot e_i) = \text{str-}\lim_i \varphi(m \cdot e_i) = \overline{\varphi}(m)$$

for all $m \in M(A)$. \square

Let X be a Hilbert $A - A$ pro- C^* -bimodule. For a closed two sided ideal \mathcal{I} of A we put $\mathcal{I}X = \text{span}\{ax/a \in \mathcal{I}, x \in X\}$ and $X\mathcal{I} = \text{span}\{xa/a \in \mathcal{I}, x \in X\}$. By [JZ2, Lemma 3.7], $\mathcal{I}X$ and $X\mathcal{I}$ are closed Hilbert pro- C^* -sub-bimodules of X .

Definition 3.17. Let (X, A) and (Y, C) be two Hilbert pro- C^* -bimodules. We say that (Y, C) is an extension of (X, A) if the following conditions are satisfied:

- (1) C contains A as an ideal;
- (2) there exists a morphism (φ_X, φ_A) of Hilbert pro- C^* -bimodules from (X, A) to (Y, C) , such that $\varphi_A : A \rightarrow C$ is just the inclusion map;
- (3) $\varphi_X(X) = \varphi_A(A)Y = Y\varphi_A(A)$.

Remark 3.18. If (Y, C) is an extension of (X, A) , and if the topology on C is given by the family of C^* -seminorms $\{p_\lambda; \lambda \in \Lambda\}$, then the topology on A is given by $\{p_\lambda|_A; \lambda \in \Lambda\}$, and $p_\lambda(\varphi_A(a)) = p_\lambda(a)$ for all $a \in A$ and for all $\lambda \in \Lambda$. Therefore, $p_\lambda^C(\varphi_X(x)) = p_\lambda^A(x)$ for all $x \in X$ and for all $\lambda \in \Lambda$, and so, for each $\lambda \in \Lambda$, there is a linear map $\varphi_{X_\lambda} : X_\lambda \rightarrow Y_\lambda$ such that $\sigma_\lambda^Y \circ \varphi_X = \varphi_{X_\lambda} \circ \sigma_\lambda^X$. Then $\varphi_X = \lim_{\leftarrow \lambda} \varphi_{X_\lambda}$, and for each $\lambda \in \Lambda$, (Y_λ, C_λ) is an extension of (X_λ, A_λ) via the morphism $(\varphi_{X_\lambda}, \varphi_{A_\lambda})$, where φ_{A_λ} is the inclusion of A_λ into C_λ .

In the following proposition, we show that $(M(X), M(A))$ is a maximal extension of (X, A) in the sense that if (Y, C) is another extension of (X, A) via a morphism (ψ_X, ψ_A) , then there is a morphism of Hilbert pro- C^* -bimodules $(\vartheta_Y, \vartheta_C) : (Y, C) \rightarrow (M(X), M(A))$ such that $\vartheta_Y \circ \psi_X = \iota_X$ and $\vartheta_C \circ \psi_A = \iota_A$ (see, for the case of Hilbert C^* -modules, [BG1, BG2]).

Proposition 3.19. Let X be a full Hilbert pro- C^* -bimodule over A . Then $(M(X), M(A))$ is a maximal extension of (X, A) .

Proof. Let (ι_X, ι_A) be the morphism of Theorem 3.13(2) between (X, A) and $(M(X), M(A))$, where $\iota_X(x)(a) = xa$, $\iota_A(a)(b) = ab$, for $x \in X$, $a, b \in A$. From [R, Corollary 3.3] we have that for every $\lambda \in \Lambda$, $M(X_\lambda)\iota_{A_\lambda}(A_\lambda) = \iota_{X_\lambda}(X_\lambda) = \iota_{A_\lambda}(A_\lambda)M(X_\lambda)$. Therefore, since from Theorem 3.13, we have that $M(X) = \lim_{\leftarrow \lambda} M(X_\lambda)$, $\iota_X = \lim_{\leftarrow \lambda} \iota_{X_\lambda}$, $\iota_A = \lim_{\leftarrow \lambda} \iota_{A_\lambda}$, and since both $\iota_A(A)M(X)$, $M(X)\iota_A(A)$ and $\iota_X(X)$ are closed submodules of $M(X)$, we deduce that $\iota_A(A)M(X) = \iota_X(X) = M(X)\iota_A(A)$. Hence $(M(X), M(A))$ is an extension of (X, A) .

To show that $(M(X), M(A))$ is a maximal extension, let (Y, C) be another extension of (X, A) via a morphism (ψ_X, ψ_A) . Then, by Remark 3.18, $\psi_X = \lim_{\leftarrow \lambda} \psi_{X_\lambda}$, $\psi_A = \lim_{\leftarrow \lambda} \psi_{A_\lambda}$, and for each $\lambda \in \Lambda$, (Y_λ, C_λ) is an extension of (X_λ, A_λ) via the morphism $(\psi_{X_\lambda}, \psi_{A_\lambda})$. By [R, Proposition 3.4], there exists a unique morphism $(\vartheta_{Y_\lambda}, \vartheta_{C_\lambda}) : (Y_\lambda, C_\lambda) \rightarrow (M(X_\lambda), M(A_\lambda))$ such that $\vartheta_{Y_\lambda} \circ \psi_{X_\lambda} = \iota_{X_\lambda}$ and $\vartheta_{C_\lambda} \circ \psi_{A_\lambda} = \iota_{A_\lambda}$. Moreover,

$$\vartheta_{Y_\lambda}(\sigma_\lambda^Y(y))(\pi_\lambda^A(a)) = \psi_{X_\lambda}^{-1}(\sigma_\lambda^Y(y)\psi_{A_\lambda}(\pi_\lambda^A(a)))$$

and

$$\vartheta_{C_\lambda}(\pi_\lambda^C(c))(\pi_\lambda^A(a)) = \psi_{A_\lambda}^{-1}(\pi_\lambda^C(c)\psi_{A_\lambda}(\pi_\lambda^A(a)))$$

for all $a \in A$, for all $c \in C$ and for all $y \in Y$. It is easy to check that $(\vartheta_{Y_\lambda})_\lambda$ is an inverse system of linear maps, $(\vartheta_{C_\lambda})_\lambda$ is an inverse system of C^* -morphisms, and $(\vartheta_Y, \vartheta_C) : (Y, C) \rightarrow (M(X), M(A))$, where $\vartheta_Y = \lim_{\leftarrow \lambda} \vartheta_{Y_\lambda}$ and $\vartheta_C = \lim_{\leftarrow \lambda} \vartheta_{C_\lambda}$, is a

morphism of Hilbert pro- C^* -bimodules such that $\vartheta_Y \circ \psi_X = \iota_X$ and $\vartheta_C \circ \psi_A = \iota_A$. \square

4. CROSSED PRODUCTS BY HILBERT PRO- C^* -MODULES

A *covariant representation* of a Hilbert pro- C^* -bimodule (X, A) on a pro- C^* -algebra B is a morphism of Hilbert pro- C^* -bimodules from (X, A) to the Hilbert pro- C^* -bimodule (B, B) .

The *crossed product* of A by a Hilbert pro- C^* -bimodule (X, A) is a pro- C^* -algebra, denoted by $A \times_X \mathbb{Z}$, and a covariant representation (i_X, i_A) of (X, A) on $A \times_X \mathbb{Z}$ with the property that for any covariant representation (φ_X, φ_A) of (X, A) on a pro- C^* -algebra B , there is a unique pro- C^* -morphism $\Phi : A \times_X \mathbb{Z} \rightarrow B$ such that $\Phi \circ i_X = \varphi_X$ and $\Phi \circ i_A = \varphi_A$ [JZ1, Definition 3.3].

Remark 4.1. If (Φ, φ) is a morphism of Hilbert pro- C^* -bimodules from (X, A) to (Y, B) , then $(i_Y \circ \Phi, i_B \circ \varphi)$ is a covariant representation of X on $B \times_Y \mathbb{Z}$ and by the universal property of $A \times_X \mathbb{Z}$ there is a unique pro- C^* -morphism $\Phi \times \varphi$ from $A \times_X \mathbb{Z}$ to $B \times_Y \mathbb{Z}$ such that $(\Phi \times \varphi) \circ i_A = i_B \circ \varphi$ and $(\Phi \times \varphi) \circ i_X = i_Y \circ \Phi$.

Lemma 4.2. Let (Φ, φ) be a morphism of Hilbert pro- C^* -bimodules from (X, A) to (Y, B) . If Γ and Γ' have the same index set and $\varphi = \lim_{\leftarrow \lambda} \varphi_\lambda$, then $\Phi = \lim_{\leftarrow \lambda} \Phi_\lambda$, for each $\lambda \in \Lambda$, $(\Phi_\lambda, \varphi_\lambda)$ is a morphism of Hilbert C^* -bimodules, $(\Phi_\lambda \times \varphi_\lambda)_\lambda$ is an inverse system of C^* -morphisms and $\Phi \times \varphi = \lim_{\leftarrow \lambda} \Phi_\lambda \times \varphi_\lambda$. Moreover, if (Φ, φ) is an isomorphism of Hilbert pro- C^* -bimodules and $\varphi_\lambda, \lambda \in \Lambda$ are C^* -isomorphisms, then $(\Phi_\lambda, \varphi_\lambda), \lambda \in \Lambda$ are isomorphisms of Hilbert C^* -bimodules.

Proof. Let $\lambda \in \Lambda$. From

$$q_\lambda^B(\Phi(x))^2 = q_\lambda(\varphi(\langle x, x \rangle)) \leq p_\lambda(\langle x, x \rangle) = p_\lambda^A(x)^2$$

for all $x \in X$, we deduce that there is a linear map $\Phi_\lambda : X_\lambda \rightarrow Y_\lambda$ such that $\Phi_\lambda \circ \sigma_\lambda^X = \sigma_\lambda^Y \circ \Phi$. It is easy to verify that $(\Phi_\lambda)_\lambda$ is an inverse system of linear maps and $\Phi = \lim_{\leftarrow \lambda} \Phi_\lambda$. Moreover, for each $\lambda \in \Lambda$, $(\Phi_\lambda, \varphi_\lambda)$ is a morphism of Hilbert C^* -bimodules. Let $\Phi_\lambda \times \varphi_\lambda$ be the C^* -morphism from $A_\lambda \times_{X_\lambda} \mathbb{Z}$ to $B_\lambda \times_{Y_\lambda} \mathbb{Z}$ induced by $(\Phi_\lambda, \varphi_\lambda)$. From

$$\begin{aligned} \pi_{\lambda\mu}^{B \times_Y \mathbb{Z}} \circ (\Phi_\lambda \times \varphi_\lambda) \circ i_{A_\lambda} &= \pi_{\lambda\mu}^{B \times_Y \mathbb{Z}} \circ i_{B_\lambda} \circ \varphi_\lambda = i_{B_\mu} \circ \pi_{\lambda\mu}^B \circ \varphi_\lambda \\ &= i_{B_\mu} \circ \varphi_\mu \circ \pi_{\lambda\mu}^A = (\Phi_\mu \times \varphi_\mu) \circ \pi_{\lambda\mu}^{A \times_X \mathbb{Z}} \circ i_{A_\lambda} \end{aligned}$$

and

$$\pi_{\lambda\mu}^{B \times_Y \mathbb{Z}} \circ (\Phi_\lambda \times \varphi_\lambda) \circ i_{X_\lambda} = (\Phi_\mu \times \varphi_\mu) \circ \pi_{\lambda\mu}^{A \times_X \mathbb{Z}} \circ i_{X_\lambda}$$

for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$ and taking into account $i_{A_\lambda}(A_\lambda)$ and $i_{X_\lambda}(X_\lambda)$ generate $A_\lambda \times_{X_\lambda} \mathbb{Z}$, we deduce that $(\Phi_\lambda \times \varphi_\lambda)_\lambda$ is an inverse system of C^* -morphisms. Moreover, since

$$\lim_{\leftarrow \lambda} (\Phi_\lambda \times \varphi_\lambda) \circ \lim_{\leftarrow \lambda} i_{A_\lambda} = \lim_{\leftarrow \lambda} (\Phi_\lambda \times \varphi_\lambda) \circ i_{A_\lambda} = \lim_{\leftarrow \lambda} i_{B_\lambda} \circ \varphi_\lambda = \lim_{\leftarrow \lambda} i_{B_\lambda} \circ \lim_{\leftarrow \lambda} \varphi_\lambda$$

and

$$\lim_{\leftarrow \lambda} (\Phi_\lambda \times \varphi_\lambda) \circ \lim_{\leftarrow \lambda} i_{X_\lambda} = \lim_{\leftarrow \lambda} (\Phi_\lambda \times \varphi_\lambda) \circ i_{X_\lambda} = \lim_{\leftarrow \lambda} i_{Y_\lambda} \circ \Phi_\lambda = \lim_{\leftarrow \lambda} i_{Y_\lambda} \circ \lim_{\leftarrow \lambda} \Phi_\lambda,$$

we obtain $\Phi \times \varphi = \lim_{\leftarrow \lambda} \Phi_\lambda \times \varphi_\lambda$.

Suppose that (Φ, φ) is an isomorphism of Hilbert pro- C^* -bimodules and $\varphi_\lambda, \lambda \in \Lambda$ are C^* -isomorphisms. Then, since $\varphi^{-1} = \lim_{\leftarrow \lambda} \varphi_\lambda^{-1}$, by the first part of the proof, $\Phi^{-1} = \lim_{\leftarrow \lambda} \psi_\lambda$ and $(\psi_\lambda, \varphi_\lambda^{-1})$ is a morphism of Hilbert C^* -bimodules for all $\lambda \in \Lambda$. Let $\lambda \in \Lambda$. From

$$\psi_\lambda \circ \Phi_\lambda \circ \sigma_\lambda^X = \psi_\lambda \circ \sigma_\lambda^Y \circ \Phi = \sigma_\lambda^X \circ \Phi^{-1} \circ \Phi = \sigma_\lambda^X$$

and

$$\Phi_\lambda \circ \psi_\lambda \circ \sigma_\lambda^Y = \Phi_\lambda \circ \sigma_\lambda^X \circ \Phi^{-1} = \sigma_\lambda^Y \circ \Phi \circ \Phi^{-1} = \sigma_\lambda^Y$$

and taking into account that σ_λ^X and σ_λ^Y are surjective, we deduce that $\psi_\lambda = \Phi_\lambda^{-1}$. \square

The following proposition gives the relation between the crossed product of A by X and the crossed product of $M(A)$ by $M(X)$.

Proposition 4.3. *Let (X, A) be full Hilbert pro- C^* -bimodule. Then $A \times_X \mathbb{Z}$ can be embedded into $M(A) \times_{M(X)} \mathbb{Z}$.*

Proof. Let ι_A be the embedding of A in $M(A)$ and ι_X the embedding of X in $M(X)$. Then (ι_X, ι_A) is a morphism of Hilbert pro- C^* -bimodules, and since $\iota_A = \lim_{\leftarrow \lambda} \iota_{A_\lambda}$, by Lemma 4.2, $\iota_X \times \iota_A = \lim_{\leftarrow \lambda} \iota_{X_\lambda} \times \iota_{A_\lambda}$ is a pro- C^* -morphism from $A \times_X \mathbb{Z}$ to $M(A) \times_{M(X)} \mathbb{Z}$. Moreover, since

$$\begin{aligned} p_{\lambda, M(A) \times_{M(X)} \mathbb{Z}}(\iota_X \times \iota_A(c)) &= \left\| \iota_{X_\lambda} \times \iota_{A_\lambda} \left(\pi_\lambda^{A \times_X \mathbb{Z}}(c) \right) \right\|_{M(A_\lambda) \times_{M(X_\lambda)} \mathbb{Z}} \\ &\quad [\text{A, Remark 2.2}] \\ &= \left\| \pi_\lambda^{A \times_X \mathbb{Z}}(c) \right\|_{A_\lambda \times_{X_\lambda} \mathbb{Z}} = p_{\lambda, A \times_X \mathbb{Z}}(c) \end{aligned}$$

for all $c \in A \times_X \mathbb{Z}$ and for all $\lambda \in \Lambda$, $A \times_X \mathbb{Z}$ can be identified with a pro- C^* -subalgebra of $M(A) \times_{M(X)} \mathbb{Z}$. \square

The following proposition is a generalization of [R, Proposition 4.7].

Proposition 4.4. *Let (X, A) be a full Hilbert pro- C^* -bimodule. Then $M(A) \times_{M(X)} \mathbb{Z}$ can be identified with a pro- C^* -subalgebra of $M(A \times_X \mathbb{Z})$.*

Proof. Since X is full, (i_X, i_A) is nondegenerate and $i_A = \lim_{\leftarrow \lambda} i_{A_\lambda}$ and $i_X = \lim_{\leftarrow \lambda} i_{X_\lambda}$ [JZ1, Propositions 3.4 and 3.5]. Then, by Theorem 3.16, (i_X, i_A) extends to a co-variant representation $(\overline{i_X}, \overline{i_A})$ of $(M(X), M(A))$ on $M(A \times_X \mathbb{Z})$, and moreover, $\overline{i_A} = \lim_{\leftarrow \lambda} \overline{i_{A_\lambda}}$ and $\overline{i_X} = \lim_{\leftarrow \lambda} \overline{i_{X_\lambda}}$. It is easy to check $(\overline{i_{X_\lambda}}, \overline{i_{A_\lambda}})$ is a covariant representation of $(M(X_\lambda), M(A_\lambda))$ on $M(A_\lambda \times_{X_\lambda} \mathbb{Z})$ for each $\lambda \in \Lambda$. By [R, Proposition 4.7], for each $\lambda \in \Lambda$, there is an injective C^* -morphism $\Phi_\lambda : M(A_\lambda) \times_{M(X_\lambda)} \mathbb{Z} \rightarrow M(A_\lambda \times_{X_\lambda} \mathbb{Z})$ such that $\Phi_\lambda \circ i_{M(X_\lambda)} = \overline{i_{X_\lambda}}$ and $\Phi_\lambda \circ i_{M(A_\lambda)} = \overline{i_{A_\lambda}}$. From

$$\begin{aligned} \pi_{\lambda\mu}^{M(A \times_X \mathbb{Z})} \circ \Phi_\lambda \circ i_{M(X_\lambda)} &= \pi_{\lambda\mu}^{M(A \times_X \mathbb{Z})} \circ \overline{i_{X_\lambda}} = \overline{i_{X_\mu}} \circ \chi_{\lambda\mu}^{M(X)} \\ &= \Phi_\mu \circ i_{M(X_\mu)} \circ \chi_{\lambda\mu}^{M(X)} = \Phi_\mu \circ \pi_{\lambda\mu}^{M(A \times_X \mathbb{Z})} \circ i_{M(X_\lambda)} \end{aligned}$$

and

$$\begin{aligned} \pi_{\lambda\mu}^{M(A \times_X \mathbb{Z})} \circ \Phi_\lambda \circ i_{M(A_\lambda)} &= \pi_{\lambda\mu}^{M(A \times_X \mathbb{Z})} \circ \overline{i_{A_\lambda}} = \overline{i_{A_\mu}} \circ \pi_{\lambda\mu}^{M(A)} \\ &= \Phi_\mu \circ i_{M(A_\mu)} \circ \pi_{\lambda\mu}^{M(A)} = \Phi_\mu \circ \pi_{\lambda\mu}^{M(A \times_X \mathbb{Z})} \circ i_{M(A_\lambda)} \end{aligned}$$

for all $\lambda, \mu \in \Lambda$, with $\lambda \geq \mu$, and taking into account that $i_{M(X_\lambda)}(M(X_\lambda))$ and $i_{M(A_\lambda)}(M(A_\lambda))$ generate $M(A_\lambda) \times_{M(X_\lambda)} \mathbb{Z}$, we deduce that $(\Phi_\lambda)_\lambda$ is an inverse system of isometric C^* -morphisms, and then $\Phi = \lim_{\leftarrow \lambda} \Phi_\lambda$ is an injective pro- C^* -morphism from $\lim_{\leftarrow \lambda} M(A_\lambda) \times_{M(X_\lambda)} \mathbb{Z}$ to $\lim_{\leftarrow \lambda} M(A_\lambda \times_{X_\lambda} \mathbb{Z})$ such that $p_{\lambda, M(A \times_X \mathbb{Z})}(\Phi(c)) = p_{\lambda, M(A) \times_{M(X)} \mathbb{Z}}(c)$ for all $c \in M(A) \times_{M(X)} \mathbb{Z}$ and for all $\lambda \in \Lambda$. Therefore, $M(A) \times_{M(X)} \mathbb{Z}$ can be identified with a pro- C^* -subalgebra of $M(A \times_X \mathbb{Z})$. \square

An automorphism α of a pro- C^* -algebra A such that $p_\lambda(\alpha(a)) = p_\lambda(a)$ for all $a \in A$ and $\lambda \in \Lambda'$, where Λ' is a cofinal subset of Λ , is called an inverse limit automorphism. If α is an inverse limit automorphism of the pro- C^* -algebra A , then $X_\alpha = \{\xi_x; x \in A\}$ is a Hilbert $A - A$ pro- C^* -bimodule with the bimodule structure defined as $\xi_x a = \xi_{xa}$, respectively $a \xi_x = \xi_{\alpha^{-1}(a)x}$, and the inner products are defined as $\langle \xi_x, \xi_y \rangle_A = x^* y$, respectively ${}_A \langle \xi_x, \xi_y \rangle = \alpha(xy^*)$. The crossed product $A \rtimes_\alpha \mathbb{Z}$ of A by α is isomorphic to the crossed product of A by X_α [JZ1].

Corollary 4.5. *If α is an inverse limit automorphism of a non unital pro- C^* -algebra A , then $M(A) \rtimes_{\overline{\alpha}} \mathbb{Z}$ can be identified with a pro- C^* -subalgebra of $M(A \rtimes_\alpha \mathbb{Z})$.*

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